Tuesday, April 26, 2016 7:52 AM

Functions, Limits and continuity

Def (Real-valued and vector-valued functions)

A function is a rule that assigns to each element in the domain an element in the range.

A function whose domain is a subset $U \subset \mathbb{R}^m$, $m \geq 1$ and whose range is contained in $\mathbb{R}^n$ is called a real valued function of m variables if $n = 1$ and a vector valued function of m variables if $n > 1$.

- A real valued function is usually written as $y = f(x_1, \ldots, x_m)$.

A vector valued function F is usually given by :

$$F(x_1, \ldots, x_m) = (F_1(x_1, \ldots, x_m), F_2(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m))$$

The function F_i's are called components (or component functions of F).

Example (Dot product and Cross product)

Let $\vec{x} = (x_1, y_1, z_1)$ and $\vec{y} = (x_2, y_2, z_2)$ be two 3-dimensional vectors.

Then the dot product of $\vec{x}$ and $\vec{y}$ is the number $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

In terms of coordinates, it can be viewed as a function from $\mathbb{R}^6$ to $\mathbb{R}$

$$f(x_1, y_1, z_1, x_2, y_2, z_2) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

The cross product can be viewed as a vector valued function from $\mathbb{R}^6$ to $\mathbb{R}^3$ given by :

$$F(x_1, y_1, z_1, x_2, y_2, z_2) = (x_2 y_3 - y_3 x_2,\ x_3 y_1 - y_1 x_3,\ x_1 y_2 - y_2 x_1)$$

Remember $\vec{x} \times \vec{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

Def A function $F : U \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is called a vector field on U.

Def A graph of a real valued function $f : U \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ is the set

$$\{ (x_1, \dots, x_m, y) \mid y = f(x_1, \dots, x_m),\ (x_1, \dots, x_m) \in U \} \subseteq \mathbb{R}^{m+1}$$

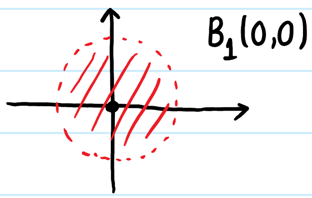
Limits and continuity

DEF An open ball centered at $\vec{a}$ in $\mathbb{R}^n$ of radius R,

$$B_R(\vec{a}) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < R \}$$

Ex What is $B_1(0,0)$?

It consists of points $(x,y) \in \mathbb{R}^2$ s.t $(x-0)^2 + (y-0)^2 \leq 1 \Rightarrow x^2 + y^2 < 1$

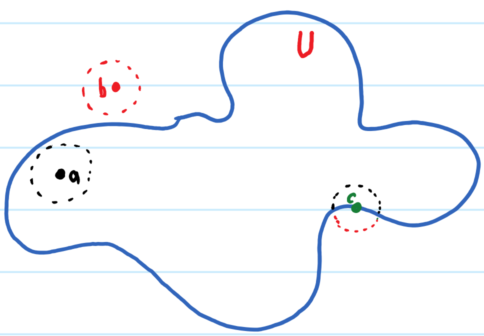


DEF A point $\vec{a}$ in $\mathbb{R}^n$ is called an <u>interior point</u> of a set $U \subset \mathbb{R}^n$, if there is an open ball centered at $\vec{a}$ that is completely contained in U.

DEF A point $\vec{a}$ in $\mathbb{R}^n$ is called an <u>exterior point</u> of set $U \subset \mathbb{R}^n$, if there is an open ball centered at $\vec{a}$ that is completely outside U.

DEF If every open ball centered at $\vec{a} \in \mathbb{R}^n$ consists of points in U and points outside of U, then we call $\vec{a}$ a <u>boundary point</u> of U.

Ex



a is an interior point,
b is an exterior point & c is a boundary point

"ε-δ" definition of limit :

Let f be defined on a subset D of $\mathbb{R}^n$.

$\lim_{\vec{x}\to\vec{a}} f(x) = L$ means that for every $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that if $\vec{x} \in D$ and $0 < \|\vec{x} - \vec{a}\| < \delta$ then $\|f(\vec{x}) - L\| < \varepsilon$.

Prop If the limit $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ holds, the limit of f along any path in U approaching $\vec{a}$ must be L. (Analogous to limit from left and right in a single variable).

Ex Compute $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$

Let's first approach the point $(0,0)$ along the x-axis.

Then $y = 0$ which gives $f(x,0) = \dfrac{x^4}{x^2} = x^2$ (for $x \neq 0$)

Then, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the axis.

Next approach $(0,0)$ along the y-axis.

So $x = 0$, $f(0,y) = \dfrac{-4y^2}{2y^2} = -2$ (for $y \neq 0$)

So $f(x,y) \to -2$ as $(x,y) \to (0,0)$ along the y-axis.

Since f has two different limits along two different lines, the given limit does not exist.

DEF (Limit of a vector valued functions)

Let $F(\vec{x}) = (F_1(\vec{x}), \dots, F_n(\vec{x}))$ and $\vec{L} = (L_1, L_2, \dots, L_n)$

We say that $F(\vec{x})$ has limit $\vec{L}$ as $\vec{x}$ approaches $\vec{a}$

$$\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = \vec{L} \quad \text{if and only if}$$

$$\lim_{\vec{x}\to\vec{a}} F_1(\vec{x}) = L_1, \dots, \lim_{\vec{x}\to\vec{a}} F_n(\vec{x}) = L_n \, .$$

Limit Laws

If the limits of $f, g : \mathbb{R}^m \longrightarrow \mathbb{R}$, $F, G : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ exist as $\vec{x}$ approaches $\vec{a}$ then

1) $\lim\limits_{\vec{x}\to\vec{a}} (F(\vec{x}) \pm G(\vec{x})) = \lim\limits_{\vec{x}\to\vec{a}} F(\vec{x}) \pm \lim\limits_{\vec{x}\to\vec{a}} G(\vec{x})$

2) $\lim\limits_{x\to\vec{a}} (f(\vec{x})g(\vec{x})) = \left(\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x})\right)\left(\lim\limits_{\vec{x}\to\vec{a}} g(\vec{x})\right)$

3) $\lim\limits_{\vec{x}\to\vec{a}} cF(\vec{x}) = c\lim\limits_{\vec{x}\to\vec{a}} F(\vec{x})$

4) If $\lim\limits_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$, then

$$\lim_{\vec{x}\to\vec{a}} \frac{f(x)}{g(x)} = \frac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}-\vec{a}} g(\vec{x})}$$

DEF A function $\vec{F} : U \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is continuous at $\vec{a} \in U$ if and only if $\lim\limits_{\vec{x}\to\vec{a}} F(\vec{x}) = F(\vec{a})$.

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Derivatives

Let f be a function of m variables $x_1, \ldots, x_m$ defined on an open set U. The partial derivative of f with respect to x_i is defined as

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h} = \frac{\partial f}{\partial x_i}(\vec{x})$$

For a function of f two variables $f(x,y)$

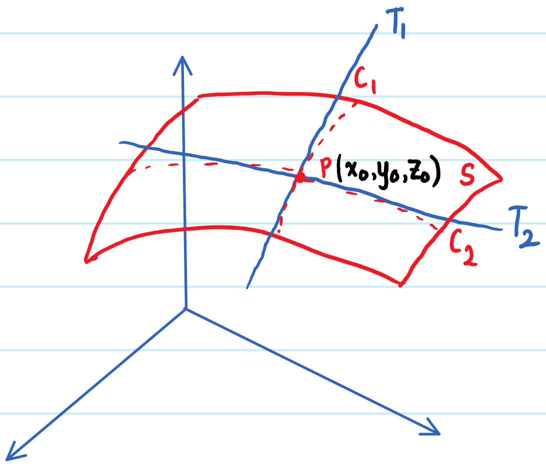
$$\frac{\partial f}{\partial x} = f_x = D_1 f = D_x f \qquad = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

<u>Remark</u> One can compute partial derivatives wrt to one variable by regarding the other variables as constant (since we fix the values of all other variables in the limit above).

<u>Ex</u> $f(x,y,z) = e^{xy} \sin(y^2 + z^2)$

$$\frac{\partial f}{\partial y} = xe^{xy} \sin(y^2 + z^2) + e^{xy} \cos(y^2 + z^2) \cdot 2y$$

Interpretation of partial derivatives



- Spse a surface S has an equation $z = f(x,y)$, and say $f(x_0,y_0) = z_0$ ie (x_0, y_0, z_0).

- let C_1 be the curve on S w/ $y = y_0$. (so the curve C_1 is where the vertical plane $y = y_0$ intersects S).

- Similarly, C_2 be the curve where the plane $x = x_0$ intersects S.

- Note that C_1 is the graph of the function $g(x) = f(x, y_0)$, so the slope of the tangent T_1 at P is $g'(x_0) = f_x(x_0, y_0)$.

- Similar for C_2.

- Partial derivatives can be interpreted as rates of change. If $z = f(x,y)$, then $\frac{\partial f}{\partial x}$ represents the rate of change wrt x when z is fixed.

Tangent plane

One of the most important idea in single variable Calculus is that if we zoom into a point on a graph of a differentiable function, the graph becomes indistinguishable from it's tangent line and therefore we can approximate the function by a linear function.

The similar idea in 3-dim'l picture is that if we zoom in towards a point on the surface, the graph of a diff function of two variables, the surface looks more and more like a plane (it's tangent plane).

<u>Def</u> Suppose a surface has equation $z = f(x,y)$, where f has continuous partial derivatives.

The the tangent plane to the surface at point P is defined to be the <u>plane</u> that contains both Tangent lines T_1 and T_2.

<u>Recall</u> A line in space is determined by a point and a direction. For instance, a line passing through point $\vec{r_0}$ and parallel to $\vec{v}$ is given by $\vec{r} = \vec{r_0} + t\vec{v}$

For a plane we Know that a plane in space is determined by a point $P_0(x_0,y_0,z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane.

Let $P(x,y,z)$ be an arbitrary point in the plane, and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of $\vec{P_0}$ and $\vec{P}$.

Then the vector $\vec{r} - \vec{r_0}$ (represented by $\overrightarrow{P_0P}$) is orthogonal to $\vec{n}$ so we have $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$ or if we write $\vec{n}$ as $\langle a,b,c \rangle$, we get

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

Now back to the tangent plane, we Know that plane passing through $P(x_0,y_0,z_0)$ is given by

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$\Rightarrow \quad z - z_0 = a(x-x_0) + b(y-y_0)$$

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If the equation represents the tangent plane, then it's intersection w/ plane $y = y_0$ must be the tangent line T_1.

Setting $y = y_0$, we get

$z - z_0 = a(x - x_0)$ and we recognize it's slope is a. And since we know the slope of line T_1 is $f_x(x_0, y_0)$, we conclude that $a = f_x(x_0, y_0)$.

Similar reasoning shows that $b = f_y(x_0, y_0)$ and therefore the equation of the tangent plane is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

<u>Def</u> The gradient of a function $f(x_1, \ldots, x_m)$ is the vector

$$\nabla f(\vec{x}) = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} \right]$$

15.6 Surface Area.

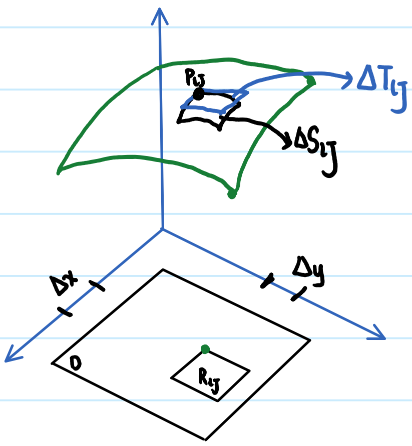
If f is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating $y = f(x)$, $a \leq x \leq b$, about the x-axis.

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx}$$

We want to compute area of a surface w/ eq^n $z = f(x,y)$, the graph of a func. of two variables.

$S \equiv$ surface w/ eqn $z = f(x,y)$, where f has continuous partial derivatives.

To simplify derivation, assume $f(x,y) \geq 0$ and domain D of f is a rectangle.



Divide D into small rectangles R_{ij} w/ area $\Delta A = \Delta x\, \Delta y$.

Let (x_i, y_j) be the corner of the rectangle closest to the origin.

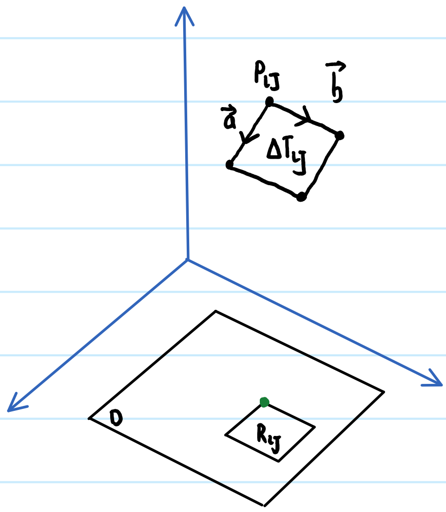
Let $P_{ij}\,(x_i, y_j, f(x_i, y_j))$ be the point closest on S directly above point (x_i, y_j).

- The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that directly lies above R_{ij}.

Then, $\sum\sum T_{ij}$ is an approximation of the total area of S and

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$$

We need a more convenient fmla :



- Let $\vec{a}$ and $\vec{b}$ be the vectors that start at P_{ij}, and lie along the sides of the parallelogram w/ area ΔT_{ij}.
- Then $\Delta T_{ij} = |\vec{a} \times \vec{b}|$.

Note that $f_x(x,y)$ and $f_y(x,y)$ are the slopes of the tangent lines through P_{ij} in the direction of $\vec{a}$ and $\vec{b}$.

Therefore,
$$\vec{a} = \Delta x\,\hat{i} + f_x(x_i,y_j)\,\Delta x\,\hat{k}$$
$$\vec{b} = \Delta y\,\hat{j} + f_y(x_i,y_j)\,\Delta y\,\hat{k}.$$

and
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x(x_i,y_j)\Delta x \\ 0 & \Delta y & f_y(x_i,y_j)\Delta y \end{vmatrix} = \left[-f_x(x_i,y_j)\,\hat{i} - f_y(x_i,y_j)\,\hat{j} + \hat{k}\right]\Delta A$$

and therefore
$$\Delta T_{ij} = |\vec{a}\times\vec{b}| = \sqrt{[f_x(x_i,y_j)]^2 + [f_y(x_i,y_j)]^2 + 1}\;\;\Delta A$$

Then,
$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{[f_x(x_i,y_j)]^2 + [f_y(x_i,y_j)]^2 + 1}\;\;\Delta A$$

and then using the definition of a double integral we get the following formula :

The area of the surface with equation $z = f(x,y)$, $(x,y) \in D$, where f_x and f_y are continuous is :

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\; dA$$

Remark Similar to formula for arc-length $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Ex Find the area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Soln $z = f(x,y) = y^2 - x^2$ with $1 \leq x^2 + y^2 \leq 4$

$$f_x(x,y) = -2x$$
$$f_y(x,y) = 2y$$

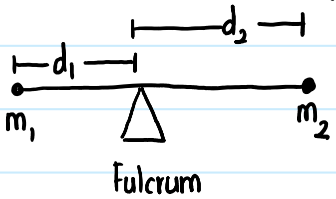
Then $A(S) = \iint_D \sqrt{4x^2 + 4y^2 + 1}\; dA$

$$= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\; r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr = 2\pi \int_5^{17} \frac{1}{8} u^{1/2}\, du = \frac{2\pi}{8}\left[\frac{2}{3}(17)^{3/2} - \frac{2}{3}(5)^{3/2}\right]$$

$$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).$$

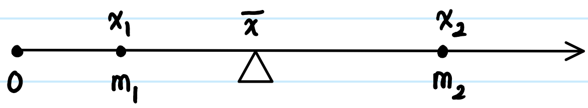
Moments and Center of mass



Let's say two masses are attached to a bar of negligible mass on the opposite sides of the fulcrum, and at distances d_1 and d_2 from the Fulcrum.

Archimedes show that the rod will balance if $\boxed{m_1 d_1 = m_2 d_2}$

- Now let's say that rod lies on the x-axis with m_1 at x_1 and m_2 at x_2 and center of mass.



Then we see that $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \Rightarrow \bar{x} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

- $m_1 x_1$ & $m_2 x_2$ are called moments of masses m_1 and m_2 (wrt 0) and the above equation shows that center of mass $\bar{x}$ is obtained by adding the moment of masses and dividing by the total mass $m = m_1 + m_2$.

Now, if we have a system of n particles w/ masses $m_1, \dots, m_n$ located at points $x_1, \dots, x_n$ on the x-axis, we can show that the center of mass of the system is located at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \quad \text{and} \quad M = \sum_{i=1}^{n} m_i x_i$$

is called the moment of the system about the origin.

Now we consider a system of n particles w/ masses $m_1, m_2, \dots, m_n$ located at the points $(x_1, y_1), \dots, (x_n, y_n)$ in the xy-plane

The moment of the system about the y-axis is defined to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the moment of the system about the x-axis as

$$M_x = \sum_{i=1}^{n} m_i y_i$$

Then the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given by the formula

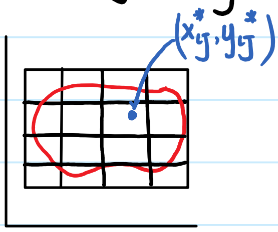
$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}, \text{ where } m = \sum m_i$$

- The center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments as the system.

- Now consider a lamina (thin plate) that occupies a region D of the xy-plane and it's density (in unit mass per unit area) is given by $\rho(x,y)$, where ρ is a continuous function.

This means that $\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$ where Δm and ΔA are the mass and area of a small rectangle that contains (x,y) and limit is taken as dimension of rectangle approach 0.

To find the total mass of the lamina, divide a rectangle R containing D into subrectangles R_{ij} and $\rho(x,y) = 0$ outside of D.



If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately

$\rho(x_i^*, y_j^*)\,\Delta A_{ij}$

Then if we add all the masses we get

$$m \approx \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_i^*, y_j^*)\,\Delta A$$

and if we take smaller and smaller rectangles we see that

$$\boxed{m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_i^*, y_j^*)\,\Delta A = \iint_D \rho(x,y)\,dA.}$$

- Now the mass of R_{ij} is approximately $\rho(x_i^*, y_j^*)\,\Delta A$, so we can approximate the moment of R_{ij} wrt to the x-axis as

$[\rho(x_i^*, y_j^*)\,\Delta A_{ij}] \cdot y_{ij}^*$ and add these all up and take limit of smaller and smaller rectangles to obtain the moment of the entire lamina about the x-axis :

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} y_{ij}^* \rho(x_i^*, y_j^*)\,\Delta A = \iint_D y\,\rho(x,y)\,dA$$

Similarly,

$$M_y = \iint_D x\,\rho(x,y)\,dA.$$

Then we can deduce that the center of mass $(\bar{x},\bar{y})$ of a lamina occupying the region D and having density function $\rho(x,y)$ are :

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} \;;\quad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} \;;$$

Moment of inertia

The moment of inertia (also called the second moment) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis.

Then we can use the ideas from before to conclude that

$$I_x = \iint_D y^2 \rho(x,y)\, dA \quad ; \quad I_y = \iint_D x^2 \rho(x,y)\, dA$$

Moment of inertia about the origin

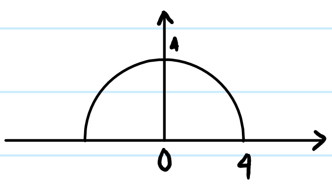
$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA$$

In particular $I_0 = I_x + I_y$

(Idea: Determines the torque needed for a desired angular acceleration)

<u>Ex</u> The density at any point on a semicircular lamina of radius 6 is twice the distance from the center of the circle. Find the center of mass of the lamina.

<u>Solution</u> Place the lamina as the upper half of the circle $x^2+y^2 = 16$.



The distance from a point (x,y) to the center of the circle to the origin is $\sqrt{x^2+y^2}$.

Then $\rho(x,y) = \sqrt{x^2+y^2}$

Then,

$$m = \iint_D 2\sqrt{x^2+y^2}\, dA$$

$$= \int_0^{\pi}\int_0^4 2r\, r\, dr\, d\theta = \int_0^{\pi} d\theta \int_0^4 2r^2 dr = \pi \frac{4^3}{3} = \frac{64\pi}{3}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\, dA = \frac{3}{64\pi}\int_0^{\pi}\int_0^4 r\sin\theta\; 2r\; r\,dr\,d\theta$$

$$= \frac{3}{64\pi}\cdot\Big[-\cos\theta\Big]_0^{\pi}\cdot\left[\frac{r^4}{4}\right]_0^4 = \frac{3}{\cancel{64}\pi}\cdot 2\cdot\frac{\cancel{64}\cdot 4}{4} = \frac{6}{\pi}.$$

$$\bar{x} = 0.$$